

Superconformal Field Theory and SUSY N=1 KdV Hierarchy I: Vertex Operators and Yang-Baxter Equation

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Abstract

The supersymmetry invariant integrable structure of two-dimensional superconformal field theory is considered. The classical limit of the corresponding infinite family of integrals of motion (IM) coincide with the family of IM of SUSY N=1 KdV hierarchy. The quantum version of the monodromy matrix, generating quantum IM, associated with the SUSY N=1 KdV is constructed via vertex operator representation of the quantum R-matrix. The possible applications to the perturbed superconformal models are discussed.

Key words: Superconformal field theory, super-KdV, Quantum superalgebras
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1 Introduction

It is known that the superconformal field theory possesses two integrable structures [1]. In the previous paper [2] we have considered the quantum super-KdV [3]-[5] hierarchy, which give rise to the infinite number of commuting integrals of motion (IM), constructed via the generators of the superconformal algebra and therefore leading to one of these integrable structures. But this set of IM is not invariant under the supersymmetry transformation.

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Here we consider the SUSY N=1 KdV hierarchy [6], [7] which is a supersymmetric generalization of the KdV one, in the case of this model the supersymmetry generator is included in the commuting family of IM [1]. The outline of the paper is as follows. In the first part (Sec. 2) we consider the classical theory of SUSY N=1 KdV system, based on the twisted affine superalgebra $C(2)^{(2)} \simeq sl(1|2)^{(2)} \simeq osp(2|2)^{(2)}$. We introduce the supersymmetric Miura transformation and monodromy matrix associated with the corresponding L-operator. Then the auxiliary **L**-matrices are constructed, which satisfy the quadratic Poisson bracket relation. As we will show in Sec. 4 quantum counterparts of these matrices coincide with a vertex-operator-represented quantum R-matrix. The quantum version of the Miura transformation i.e. the free field representation of the superconformal algebra is given in Sec. 3. In Sec. 4 the quantum $C_q(2)^{(2)}$ superalgebra [8] is introduced. Then it is shown that the corresponding quantum R-matrix can be represented by two vertex operators, satisfying Serre relations of lower Borel subalgebra of $C_q(2)^{(2)}$ and in the classical limit it coincides with **L**-matrix. The vertex-operator-represented quantum R-matrix $\mathbf{L}^{(q)}$ satisfies the so-called RTT-relation, which give us possibility to consider the model from a point of view of Quantum Inverse Scattering Method (QISM) [9], [10]. This could be well applied (Sec.5) to the study integrable perturbed superconformal theories with supersymmetry unbroken, arising in the physics of 2D disordered systems, lattice models (e.g. the tricritical Ising model), and in the superstring physics (e.g. supersymmetric D-branes) [11]-[12].

2 Integrable SUSY N=1 KdV hierarchy

The SUSY N=1 KdV system can be constructed by means of the Drinfeld-Sokolov reduction applied to the $C(2)^{(2)}$ twisted affine superalgebra [7]. The corresponding \mathcal{L} -operator has the following form:

$$\tilde{\mathcal{L}}_F = D_{u,\theta} - D_{u,\theta}\Phi(h_1 + h_2) - \lambda(e_1^+ + e_2^+ + e_1^- - e_2^-), \quad (1)$$

where $D_{u,\theta} = \partial_\theta + \theta\partial_u$ is a superderivative, the variable u lies on a cylinder of circumference 2π , θ is a Grassmann variable, $\Phi(u, \theta) = \phi(u) - \frac{i}{\sqrt{2}}\theta\xi(u)$ is a bosonic superfield; $h_1, h_2, e_1^\pm, e_2^\pm$ are the Chevalley generators of $C(2)$ with the following commutation relations:

$$\begin{aligned} [h_1, h_2] &= 0, & [h_1, e_2^\pm] &= \pm e_2^\pm, & [h_2, e_1^\pm] &= \pm e_1^\pm, \\ ad_{e_1^\pm}^2 e_2^\pm &= 0, & ad_{e_2^\pm}^2 e_1^\pm &= 0, \\ [h_\alpha, e_\alpha^\pm] &= 0 \quad (\alpha = 1, 2), & [e_\beta^\pm, e_{\beta'}^\mp] &= \delta_{\beta,\beta'} h_\beta \quad (\beta, \beta' = 1, 2), \end{aligned} \quad (2)$$

where the supercommutator $[,]$ is defined as follows: $[a, b] \equiv ad_a b \equiv ab - (-1)^{p(a)p(b)}ba$, where parity p is equal to 1 for odd elements and is equal to 0 for even ones. In the particular case of $C(2)$ $h_{1,2}$ are even and $e_{1,2}^\pm$ are odd. The operator (1) can be considered as more general one, taken in the evaluation representation of $C(2)^{(2)}$:

$$\mathcal{L}_F = D_{u,\theta} - D_{u,\theta}\Phi h_\alpha - (e_{\delta-\alpha} + e_\alpha), \quad (3)$$

where h_α , $e_{\delta-\alpha}$, e_α are the Chevalley generators of $C(2)^{(2)}$ with such commutation relations:

$$\begin{aligned} [h_{\alpha_1}, h_{\alpha_0}] &= 0, & [h_{\alpha_0}, e_{\pm\alpha_1}] &= \mp e_{\pm\alpha_1}, & [h_{\alpha_1}, e_{\pm\alpha_0}] &= \mp e_{\pm\alpha_0}, \\ [h_{\alpha_i}, e_{\pm\alpha_i}] &= \pm e_{\pm\alpha_i}, & [e_{\pm\alpha_i}, e_{\mp\alpha_j}] &= \delta_{i,j} h_{\alpha_i}, & (i, j &= 0, 1), \\ ad_{e_{\pm\alpha_0}}^3 e_{\pm\alpha_1} &= 0, & ad_{e_{\pm\alpha_1}}^3 e_{\pm\alpha_0} &= 0 \end{aligned} \quad (4)$$

where $p(h_{\alpha_{0,1}}) = 0$, $p(e_{\pm\alpha_{0,1}}) = 1$ and $\alpha_1 \equiv \alpha$, $\alpha_0 \equiv \delta - \alpha$. The Poisson brackets for the field Φ , obtained by means of the Drinfeld-Sokolov reduction are:

$$\{D_{u,\theta}\Phi(u, \theta), D_{u',\theta'}\Phi(u', \theta')\} = D_{u,\theta}(\delta(u - u')(\theta - \theta')) \quad (5)$$

and the following boundary conditions are imposed on the components of Φ : $\phi(u + 2\pi) = \phi(u) + 2\pi ip$, $\xi(u + 2\pi) = \pm \xi(u)$. The \mathcal{L}_F -operator is written in the Miura form, making a gauge transformation one can obtain a new superfield $\mathcal{U}(u, \theta) \equiv D_{u,\theta}\Phi(u, \theta)\partial_u\Phi(u, \theta) - D_{u,\theta}^3\Phi(u, \theta) = -\theta U(u) - i\alpha(u)/\sqrt{2}$, where U and α generate the superconformal algebra under the Poisson brackets:

$$\begin{aligned} \{U(u), U(v)\} &= \delta'''(u - v) + 2U'(u)\delta(u - v) + 4U(u)\delta'(u - v), \\ \{U(u), \alpha(v)\} &= 3\alpha(u)\delta'(u - v) + \alpha'(u)\delta(u - v), \\ \{\alpha(u), \alpha(v)\} &= 2\delta''(u - v) + 2U(u)\delta(u - v). \end{aligned} \quad (6)$$

Using one of the corresponding infinite family of IM, which are in involution under the Poisson brackets (these IM could be extracted from the monodromy matrix of \mathcal{L}_B -operator, see below)[1]:

$$\begin{aligned} I_1^{(cl)} &= \frac{1}{2\pi} \int U(u) du, \\ I_3^{(cl)} &= \frac{1}{2\pi} \int (U^2(u) + \alpha(u)\alpha'(u)/2) du, \\ I_5^{(cl)} &= \frac{1}{2\pi} \int (U^3(u) - (U')^2(u)/2 - \alpha'(u)\alpha''(u)/4 - \alpha'(u)\alpha(u)U(u)) du, \\ &\vdots \end{aligned} \quad (7)$$

one can obtain an evolution equation; for example, taking I_2 we get the SUSY N=1 KdV equation [6]: $\mathcal{U}_t = -\mathcal{U}_{uuu} + 3(\mathcal{U}D_{u,\theta}\mathcal{U})_u$ and in components: $U_t = -U_{uuu} - 6UU_u - \frac{3}{2}\alpha\alpha_{uu}$, $\alpha_t = -4\alpha_{uuu} - 3(U\alpha)_u$. As we have noted in the introduction one can show that the IM are invariant under supersymmetry transformation generated by $\int_0^{2\pi} du\alpha(u)$.

In order to construct the so-called monodromy matrix we introduce the \mathcal{L}_B -operator, equivalent to the \mathcal{L}_F one:

$$\mathcal{L}_B = \partial_u - \phi'(u)h_{\alpha_1} + (e_{\alpha_1} + e_{\alpha_0} - \frac{i}{\sqrt{2}}\xi h_{\alpha_1})^2 \quad (8)$$

The equivalence can be easily established if one considers the linear problem associated with the \mathcal{L}_F -operator: $\mathcal{L}_F\chi(u, \theta) = 0$ (we consider this operator acting in some representation of $C(2)^{(2)}$ and $\chi(u, \theta)$ is the vector in this representation). Then, expressing $\chi(u, \theta)$ in components: $\chi(u, \theta) = \chi_0(u) + \theta\chi_1(u)$, we find: $\mathcal{L}_B\chi_0 = 0$ and $\chi_1 = (e_{\alpha_1} + e_{\alpha_0} - \frac{i}{\sqrt{2}}\xi h_{\alpha_1})\chi_0$.

The solution to the equation $\mathcal{L}_B\chi_0 = 0$ can be written in the following way:

$$\begin{aligned} \chi_0(u) = & e^{\phi(u)h_{\alpha_1}} P \exp \int_0^u du' \left(\frac{i}{\sqrt{2}}\xi(u')e^{-\phi(u')}e_{\alpha_1} \right. \\ & \left. - \frac{i}{\sqrt{2}}\xi(u')e^{\phi(u')}e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u')} - e_{\alpha_1}^2 e^{2\phi(u')} - [e_{\alpha_1}, e_{\alpha_0}] \right) \eta, \end{aligned} \quad (9)$$

where η is a constant vector in the corresponding representation of $C(2)^{(2)}$. Therefore we can define the monodromy matrix in the following way:

$$\begin{aligned} \mathbf{M} = & e^{2\pi i p h_{\alpha_1}} P \exp \int_0^{2\pi} du \left(\frac{i}{\sqrt{2}}\xi(u)e^{-\phi(u)}e_{\alpha_1} \right. \\ & \left. - \frac{i}{\sqrt{2}}\xi(u)e^{\phi(u)}e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u)} - e_{\alpha_1}^2 e^{2\phi(u)} - [e_{\alpha_1}, e_{\alpha_0}] \right). \end{aligned} \quad (10)$$

Introducing then (as in [14]) the auxiliary \mathbf{L} -operators: $\mathbf{L} = e^{-\pi i p h_{\alpha_1}} \mathbf{M}$ we find that in the evaluation representation (when λ , the spectral parameter appears) the following Poisson bracket relation is satisfied [15]:

$$\{\mathbf{L}(\lambda) \otimes, \mathbf{L}(\mu)\} = [\mathbf{r}(\lambda\mu^{-1}), \mathbf{L}(\lambda) \otimes \mathbf{L}(\mu)], \quad (11)$$

where $\mathbf{r}(\lambda\mu^{-1})$ is trigonometric $C(2)^{(2)}$ r-matrix [16]. From this relation one obtains that the supertraces of monodromy matrices $\mathbf{t}(\lambda) = \text{str} \mathbf{M}(\lambda)$ commute under the Poisson bracket: $\{\mathbf{t}(\lambda), \mathbf{t}(\mu)\} = 0$. Expanding $\log(\mathbf{t}(\lambda))$ in λ

in the evaluation representation corresponding to the defining 3-dimensional representation of $C(2) \pi_{1/2}$ we find:

$$\lim_{\lambda \rightarrow \infty} \log(\mathbf{t}_{1/2}(\lambda)) = \sum_{n=1}^{\infty} c_n I_{2n-1}^{(cl)} \lambda^{-4n+2}, \quad (12)$$

where $c_n = (-1)^{n-1} \frac{2^n}{n!} (2n-1)!!$. So, one can obtain the IM from the supertrace of the monodromy matrix. Using the Poisson bracket relation for these supertraces with different values of spectral parameter (see above) we find that infinite family of IM is involutive, as it was mentioned earlier.

3 Free field representation of superconformal algebra

In this section we begin to build quantum counterparts of the introduced classical objects. We will start from the quantum Miura transformation, the free field representation of the superconformal algebra [17]:

$$\begin{aligned} -\beta^2 T(u) &= : \phi'^2(u) : - (1 - \beta^2/2) \phi''(u) + \frac{1}{2} : \xi \xi'(u) : + \frac{\epsilon \beta^2}{16} \\ \frac{i^{1/2} \beta^2}{\sqrt{2}} G(u) &= \phi' \xi(u) - (1 - \beta^2/2) \xi'(u), \end{aligned} \quad (13)$$

where

$$\begin{aligned} \phi(u) &= iQ + iP u + \sum_n \frac{a_{-n}}{n} e^{inu}, \quad \xi(u) = i^{-1/2} \sum_n \xi_n e^{-inu}, \\ [Q, P] &= \frac{i}{2} \beta^2, \quad [a_n, a_m] = \frac{\beta^2}{2} n \delta_{n+m,0}, \quad \{\xi_n, \xi_m\} = \beta^2 \delta_{n+m,0}. \end{aligned} \quad (14)$$

Recall that there are two types of boundary conditions on ξ : $\xi(u+2\pi) = \pm \xi(u)$. The sign “+” corresponds to the R sector, the case when ξ is integer modded, the “−” sign corresponds to the NS sector and ξ is half-integer modded. The variable ϵ in (13) is equal to zero in the R case and equal to 1 in the NS case. One can expand $T(u)$ and $G(u)$ by modes in such a way: $T(u) = \sum_n L_{-n} e^{inu} - \frac{\hat{c}}{16}$, $G(u) = \sum_n G_{-n} e^{inu}$, where $\hat{c} = 5 - 2(\frac{\beta^2}{2} + \frac{2}{\beta^2})$ and L_n, G_m generate the superconformal algebra:

$$\begin{aligned} [L_n, L_m] &= (n-m) L_{n+m} + \frac{\hat{c}}{8} (n^3 - n) \delta_{n,-m}, \quad [L_n, G_m] = (\frac{n}{2} - m) G_{m+n} \\ [G_n, G_m] &= 2L_{n+m} + \delta_{n,-m} \frac{\hat{c}}{2} (n^2 - 1/4). \end{aligned} \quad (15)$$

In the classical limit $c \rightarrow -\infty$ (the same is $\beta^2 \rightarrow 0$) the following substitution: $T(u) \rightarrow -\frac{\hat{c}}{4}U(u)$, $G(u) \rightarrow -\frac{\hat{c}}{2\sqrt{2i}}\alpha(u)$, $[\cdot, \cdot] \rightarrow \frac{4\pi}{i\hat{c}}\{\cdot, \cdot\}$ reduce the above algebra to the Poisson bracket algebra of SUSY N=1 KdV theory.

Let now F_p be the highest weight module over the oscillator algebra of a_n, ξ_m with the highest weight vector (ground state) $|p\rangle$ determined by the eigenvalue of P and nilpotency condition of the action of the positive modes: $P|p\rangle = p|p\rangle$, $a_n|p\rangle = 0$, $\xi_m|p\rangle = 0$ where $n, m > 0$. In the case of the R sector the highest weight becomes doubly degenerate due to the presence of zero mode ξ_0 . So, there are two ground states $|p, +\rangle$ and $|p, -\rangle$: $|p, +\rangle = \xi_0|p, -\rangle$. Using the above free field representation of the superconformal algebra one can obtain that for generic \hat{c} and p , F_p is isomorphic to the super-Virasoro module with the highest weight vector $|p\rangle$: $L_0|p\rangle = \Delta_{NS}|p\rangle$, where $\Delta_{NS} = (p/\beta)^2 + (\hat{c} - 1)/16$ in the NS sector and module with two highest weight vectors in the Ramond case: $L_0|p, \pm\rangle = \Delta_R|p, \pm\rangle$, $\Delta_R = (p/\beta)^2 + \hat{c}/16$, $|p, +\rangle = (\beta^2/\sqrt{2}p)G_0|p, -\rangle$. The space F_p , now considered as super-Virasoro module, splits into the sum of finite-dimensional subspaces, determined by the value of L_0 : $F_p = \bigoplus_{k=0}^{\infty} F_p^{(k)}$, $L_0 F_p^{(k)} = (\Delta + k)F_p^{(k)}$. The quantum versions of local integrals of motion should act invariantly on the subspaces $F_p^{(k)}$. Thus, the diagonalization of IM reduces (in a given subspace $F_p^{(k)}$) to the finite purely algebraic problem, which however rapidly become rather complicated for large k . It should be noted also that in the case of the Ramond sector supersymmetry generator G_0 commute with IM, so IM act in $|p, +\rangle$ and $|p, -\rangle$ independently, without mixing of these two ground states (unlike the super-KdV case [2]).

4 Quantum monodromy matrix and RTT-relation

In this part of the work we will consider the quantum $C_q(2)^{(2)}$ R-matrix and show that the vertex operator representation of the lower Borel subalgebra of $C_q(2)^{(2)}$ allows to represent this R-matrix in the P-exponent like form which in the classical limit coincide with the auxiliary L-operator. $C_q(2)^{(2)}$ is a quantum superalgebra with the following commutation relations [8]:

$$\begin{aligned} [h_{\alpha_0}, h_{\alpha_1}] &= 0, & [h_{\alpha_0}, e_{\pm\alpha_1}] &= \mp e_{\pm\alpha_1}, & [h_{\alpha_1}, e_{\pm\alpha_0}] &= \mp e_{\pm\alpha_0}, \\ [h_{\alpha_i}, e_{\pm\alpha_i}] &= \pm e_{\pm\alpha_i} \quad (i = 0, 1), & [e_{\pm\alpha_i}, e_{\mp\alpha_j}] &= \delta_{i,j} [h_{\alpha_i}] \quad (i, j = 0, 1), \\ [e_{\pm\alpha_1}, [e_{\pm\alpha_1}, [e_{\pm\alpha_1}, e_{\pm\alpha_0}]_q]_q]_q &= 0, & [[e_{\pm\alpha_1}, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q &= 0, \end{aligned} \quad (16)$$

where $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$, $p(h_{\alpha_{0,1}}) = 0$, $p(e_{\pm\alpha_{0,1}}) = 1$ and q-supercommutator is defined in the following way: $[e_\gamma, e_{\gamma'}]_q \equiv e_\gamma e_{\gamma'} - (-1)^{p(e_\gamma)p(e_{\gamma'})} q^{(\gamma, \gamma')} e_{\gamma'} e_\gamma$, $q = e^{i\pi \frac{\beta^2}{2}}$. The corresponding coproducts are:

$$\begin{aligned}\Delta(h_{\alpha_j}) &= h_{\alpha_j} \otimes 1 + 1 \otimes h_{\alpha_j}, & \Delta(e_{\alpha_j}) &= e_{\alpha_j} \otimes q^{h_{\alpha_j}} + 1 \otimes e_{\alpha_j}, \\ \Delta(e_{-\alpha_j}) &= e_{-\alpha_j} \otimes 1 + q^{-h_{\alpha_j}} \otimes e_{-\alpha_j}.\end{aligned}\quad (17)$$

The associated R-matrix can be expressed in such a way [8]: $R = KR_+R_0R_-$, where $K = q^{h_\alpha \otimes h_\alpha}$, $R_+ = \prod_{n \geq 0} R_{n\delta + \alpha}$, $R_- = \prod_{n \geq 1} R_{n\delta - \alpha}$, $R_0 = \exp((q - q^{-1}) \sum_{n > 0} d(n) e_{n\delta} \otimes e_{-n\delta})$. Here $\bar{R}_\gamma = \exp_{(-q^{-1})}(A(\gamma)(q - q^{-1})(e_\gamma \otimes e_{-\gamma}))$ and coefficients A and d are defined as follows: $A(\gamma) = \{(-1)^n \text{ if } \gamma = n\delta + \alpha; (-1)^{n-1} \text{ if } \gamma = n\delta - \alpha\}$, $d(n) = \frac{n(q - q^{-1})}{q^n - q^{-n}}$. The generators $e_{n\delta}$, $e_{n\delta \pm \alpha}$ are defined via the q -commutators of Chevalley generators. In the following we will need the expressions only for simplest ones: $e_\delta = [e_{\alpha_0}, e_{\alpha_1}]_{q^{-1}}$ and $e_{-\delta} = [e_{-\alpha_1}, e_{-\alpha_0}]_q$. The elements $e_{n\delta \pm \alpha}$ are expressed as multiple commutators of e_δ with corresponding Chevalley generators, $e_{n\delta}$ ones have more complicated form [8].

Let's introduce the reduced R-matrix $\bar{R} \equiv K^{-1}R$. Using all previous information one can write \bar{R} as $\bar{R}(\bar{e}_{\alpha_i}, \bar{e}_{-\alpha_i})$, where $\bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1$ and $\bar{e}_{-\alpha_i} = 1 \otimes e_{-\alpha_i}$, because it is represented as power series of these elements. After this necessary background we will introduce vertex operators and using the fact that they satisfy the Serre relations of the lower Borel subalgebra of $C_q(2)^{(2)}$ we will prove that the reduced R-matrix, represented by the vertex operators has the properties of the P-exponent. So, the vertex operators are: $V_1 = \frac{1}{q^{-1}-q} \int d\theta \int_{u_2}^{u_1} du : e^{-\Phi} :$, $V_0 = \frac{1}{q^{-1}-q} \int d\theta \int_{u_2}^{u_1} du : e^{\Phi} :$, where $2\pi \geq u_1 \geq u_2 \geq 0$, $\Phi = \phi(u) - \frac{i}{\sqrt{2}}\theta\xi(u)$ is a superfield and normal ordering here means that $: e^{\pm\phi(u)} := \exp\left(\pm \sum_{n=1}^{\infty} \frac{a_{-n}}{n} e^{inu}\right) \exp\left(\pm i(Q + Pu)\right) \exp\left(\mp \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-inu}\right)$. One can show via the standard contour technique that these operators satisfy the same commutation relations as e_{α_1} , e_{α_0} correspondingly.

Then, following [18] one can show (using the fundamental property of the universal R-matrix: $(I \otimes \Delta)R = R^{13}R^{12}$) that the reduced R-matrix has the following property:

$$\bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i} + e''_{-\alpha_i}) = \bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i})\bar{R}(\bar{e}_{\alpha_i}, e''_{-\alpha_i}), \quad (18)$$

where $e'_{-\alpha_i} = 1 \otimes q^{-h_{\alpha_i}} \otimes e_{-\alpha_i}$, $e''_{-\alpha_i} = 1 \otimes e_{-\alpha_i} \otimes 1$, $\bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1 \otimes 1$. The commutation relations between them are:

$$\begin{aligned}e'_{-\alpha_i} \bar{e}_{\alpha_j} &= -\bar{e}_{\alpha_j} e'_{-\alpha_i}, & e''_{-\alpha_i} \bar{e}_{\alpha_j} &= -\bar{e}_{\alpha_j} e''_{-\alpha_i}, \\ e'_{-\alpha_i} e''_{-\alpha_j} &= -q^{b_{ij}} e''_{-\alpha_j} e'_{-\alpha_i},\end{aligned}\quad (19)$$

where b_{ij} is the symmetric matrix with the following elements: $b_{00} = b_{11} = -b_{01} = 1$. Now, denoting $\bar{\mathbf{L}}^{(q)}(u_2, u_1)$ the reduced R-matrix with $e_{-\alpha_i}$ represented by V_i , we find, using the above property of \bar{R} with $e'_{-\alpha_i}$ replaced by appropriate vertex operators: $\bar{\mathbf{L}}^{(q)}(u_3, u_1) = \bar{\mathbf{L}}^{(q)}(u_2, u_1)\bar{\mathbf{L}}^{(q)}(u_3, u_2)$ with $u_1 \geq u_2 \geq u_3$. So, $\bar{\mathbf{L}}^{(q)}$ has the property of P-exponent. But because of singularities in the operator products of vertex operators it can not be written

in the usual P-ordered form. Thus, we propose a new notion, the quantum P-exponent:

$$\bar{\mathbf{L}}^{(q)}(u_1, u_2) = P \exp^{(q)} \int_{u_2}^{u_1} du \int d\theta (e_{\alpha_1} : e^{-\Phi} : + e_{\alpha_0} : e^{\Phi} :). \quad (20)$$

Introducing new object: $\mathbf{L}^{(q)} \equiv e^{i\pi P h_{\alpha_1}} \bar{\mathbf{L}}^{(q)}(0, 2\pi)$, which coincides with R-matrix with $1 \otimes h_{\alpha_1}$ replaced by $2P/\beta^2$ and $1 \otimes e_{-\alpha_1}$, $1 \otimes e_{-\alpha_0}$ replaced by V_1 and V_0 (with integration from 0 to 2π) correspondingly, we find that it satisfies the well-known RTT-relation:

$$\begin{aligned} \mathbf{R}(\lambda \mu^{-1}) (\mathbf{L}^{(q)}(\lambda) \otimes \mathbf{I}) (\mathbf{I} \otimes \mathbf{L}^{(q)}(\mu)) \\ = (\mathbf{I} \otimes \mathbf{L}^{(q)}(\mu)) (\mathbf{L}^{(q)}(\lambda) \otimes \mathbf{I}) \mathbf{R}(\lambda \mu^{-1}), \end{aligned} \quad (21)$$

where the dependence on λ, μ means that we are considering $\mathbf{L}^{(q)}$ -operators in the evaluation representation of $C_q(2)^{(2)}$. Thus the supertraces of “shifted” $\mathbf{L}^{(q)}$ -operators, the transfer matrices $\mathbf{t}^{(q)}(\lambda) \equiv \text{str}(e^{i\pi P h_{\alpha_1}} \mathbf{L}^{(q)}(\lambda))$ commute: $[\mathbf{t}^{(q)}(\lambda), \mathbf{t}^{(q)}(\mu)] = 0$, giving the quantum integrability.

Now we will show that in the classical limit ($q \rightarrow 1$) the $\mathbf{L}^{(q)}$ -operator will give the auxiliary \mathbf{L} -matrix defined in the Sec. 2. We will use the P-exponent property of $\bar{\mathbf{L}}^{(q)}(0, 2\pi)$. Let's decompose $\bar{\mathbf{L}}^{(q)}(0, 2\pi)$ in the following way: $\bar{\mathbf{L}}^{(q)}(0, 2\pi) = \lim_{N \rightarrow \infty} \prod_{m=1}^N \bar{\mathbf{L}}^{(q)}(x_{m-1}, x_m)$, where we divided the interval $[0, 2\pi]$ into infinitesimal intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \epsilon = 2\pi/N$. Let's find the terms that can give contribution of the first order in ϵ in $\bar{\mathbf{L}}^{(q)}(x_{m-1}, x_m)$. In this analysis we will need the operator product expansion of vertex operators:

$$\begin{aligned} \xi(u) \xi(u') &= -\frac{i\beta^2}{(iu - iu')} + \sum_{k=1}^{\infty} c_k(u) (iu - iu')^k, \\ :e^{a\phi(u)}::e^{b\phi(u')} &:= (iu - iu')^{\frac{ab\beta^2}{2}} (:e^{(a+b)\phi(u)}: + \sum_{k=1}^{\infty} d_k(u) (iu - iu')^k), \end{aligned} \quad (22)$$

where $c_k(u)$ and $d_k(u)$ are operator-valued functions of u . Now one can see that only two types of terms can give the contribution of the order ϵ in $\bar{\mathbf{L}}^{(q)}(x_{m-1}, x_m)$ when $q \rightarrow 1$. The first type consists of operators of the first order in V_i and the second type is formed by the operators, quadratic in V_i , which give contribution of the order $\epsilon^{1 \pm \beta^2}$ by virtue of operator product expansion. Let's look on the terms of the second type in detail. At first we consider the terms appearing from the R_0 -part of R-matrix, represented by vertex operators:

$$\begin{aligned} & \frac{e_\delta}{2(q - q^{-1})} \left(\int_{x_{m-1}}^{x_m} du_1 : e^{-\phi} : \xi(u_1 - i0) \int_{x_{m-1}}^{x_m} du_2 : e^\phi : \xi(u_2 + i0) + \right. \\ & \left. q^{-1} \int_{x_{m-1}}^{x_m} du_2 : e^\phi : \xi(u_2 - i0) \int_{x_{m-1}}^{x_m} du_1 : e^{-\phi} : \xi(u_1 + i0) \right) \end{aligned} \quad (23)$$

Neglecting the terms, which give rise to $O(\epsilon^2)$ contribution, we obtain, using the operator products of vertex operators:

$$\begin{aligned} & \frac{e_\delta}{2(q - q^{-1})} \left(\int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{x_m} du_2 \frac{-i\beta^2}{(i(u_1 - u_2 - i0))^{\frac{\beta^2}{2}+1}} + \right. \\ & \left. q^{-1} \int_{x_{m-1}}^{x_m} du_2 \int_{x_{m-1}}^{x_m} du_1 \frac{-i\beta^2}{(i(u_2 - u_1 - i0))^{\frac{\beta^2}{2}+1}} \right) \end{aligned} \quad (24)$$

In the $\beta^2 \rightarrow 0$ limit we get: $\frac{[e_{\alpha_1}, e_{\alpha_0}]}{2i\pi} \int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{x_m} du_2 \left(\frac{1}{u_1 - u_2 + i0} - \frac{1}{u_1 - u_2 - i0} \right)$ which, by the well known formula: $\frac{1}{x+i0} = P\frac{1}{x} - i\pi\delta(x)$ gives: $-\int_{x_{m-1}}^{x_m} du [e_{\alpha_1}, e_{\alpha_0}]$. Another terms arise from the R_+ and R_- parts of R-matrix and are very similar to each other:

$$\begin{aligned} & \frac{e_{\alpha_0}^2}{2(2)_{(-q^{-1})}} \int_{x_{m-1}}^{x_m} du_1 : e^\phi : \xi(u_1 - i0) \int_{x_{m-1}}^{x_m} du_2 : e^\phi : \xi(u_2 + i0), \\ & \frac{e_{\alpha_1}^2}{2(2)_{(-q^{-1})}} \int_{x_{m-1}}^{x_m} du_1 : e^{-\phi} : \xi(u_1 - i0) \int_{x_{m-1}}^{x_m} du_2 : e^{-\phi} : \xi(u_2 + i0). \end{aligned} \quad (25)$$

The integrals can be reduced to the ordered ones:

$$\begin{aligned} & \frac{e_{\alpha_0}^2}{2} \int_{x_{m-1}}^{x_m} du_1 : e^\phi : \xi(u_1) \int_{x_{m-1}}^{u_1} du_2 : e^\phi : \xi(u_2), \\ & \frac{e_{\alpha_1}^2}{2} \int_{x_{m-1}}^{x_m} du_1 : e^{-\phi} : \xi(u_1) \int_{x_{m-1}}^{u_1} du_2 : e^{-\phi} : \xi(u_2). \end{aligned} \quad (26)$$

Following [2] we find that their contribution (of order ϵ) in the classical limit is:

$$-e_{\alpha_0}^2 \int_{x_{m-1}}^{x_m} du e^{2\phi(u)}, \quad -e_{\alpha_1}^2 \int_{x_{m-1}}^{x_m} du e^{-2\phi(u)}. \quad (27)$$

Gathering now all the terms of order ϵ we find:

$$\begin{aligned} \bar{\mathbf{L}}^{(1)}(x_{m-1}, x_m) = 1 + \int_{x_{m-1}}^{x_m} du \left(\frac{i}{\sqrt{2}} \xi(u) e^{-\phi(u)} e_{\alpha_1} - \right. \\ \left. \frac{i}{\sqrt{2}} \xi(u) e^{\phi(u)} e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u)} - e_{\alpha_1}^2 e^{2\phi(u)} - [e_{\alpha_1}, e_{\alpha_0}] \right) + O(\epsilon^2) \end{aligned} \quad (28)$$

and collecting all $\bar{\mathbf{L}}^{(1)}(x_{m-1}, x_m)$ we find that $\bar{\mathbf{L}}^{(1)} = e^{-i\pi p h_{\alpha_1}} \mathbf{L}$. Therefore $\mathbf{L}^{(1)} = \mathbf{L}$.

5 Final remarks

The obtained RTT-relation (21) and corresponding quantum integrability condition give us possibility to consider the model possessing the associated IM from a point of view of QISM [9], [10]. In our case the class of such systems is very wide. It contains all the superconformal (minimal) models and some of their perturbations (though in the unperturbed case SCFT usually provides easier methods). The perturbations that do not break the integrability (commuting with IM) and therefore appropriate for the QISM scheme are (following the arguments of [14],[19]) $\phi_{(1,3)}$ and $\phi_{(3,1)}$ operators, corresponding to the vertex operators $\int_0^{2\pi} du \int d\theta e^{\pm\Phi}$. The topic of special interest is the SCFT with perturbation on the boundary related to the D-brane theory [12], [13].

In order to find the eigenvalues of the transfer matrices (they are our main object of study, because their expansion in λ gives the quantum IM) one can follow two routes. The first one, introduced in [14], is based on the so-called “fusion relations” [10] for the transfer matrices. In the case when q is a root of unity (β^2 and c are rational), the system of fusion relations becomes finite and reduces to the Thermodynamic Bethe Ansatz equations [20].

Another approach is the Baxter Q-operator method, which could be applied for all values of β^2 (therefore for all values of the central charge)[21]. The construction of the Q-operator and fusion relations will be given in the paper under preparation [22].

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